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## How Many Days Are in a Year?

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Calendar design is a classic use of fractions, and is a good application of successive approximations and basic error analysis.

**Background:** The true length of a year on Earth is 365.2422 days, or about 365.25 days. We keep our calendar in sync with the seasons by having most years 365 days long but making just under 1/4 of all years 366-day "leap" years.

**Exercise:** Design a reasonable calendar for an imaginary planet. Your calendar will consist of a pattern of 366-day "leap" years and 365-day regular years that approximates your planet's average number of days per year.

**Presentation:** Give each student a different fractional number of days per year, perhaps using dice. You can also give each planet a whimsical name like "George-world" or "Planet Sue". Some sample year lengths might be 365.1234 days, 365.1119 days, 365.8712 days.

Then present the history of the Gregorian Calendar as a story, drawing upon material from the other sections as appropriate.

### History:

- The ancient Egyptians and others used a year with exactly 365 days.

- Julius Caesar instituted the Julian Calendar in which a year has an extra day if its number is evenly divisible by 4: 4, 8, ... 1996, 2000, 2004 ... Thus the average year has  $365 + 1/4 = 365.25$  days.

The sign of the error changed in moving from the no-leap-year calendar to the Julian system. The simple system had a positive error meaning the calendar year is too short and so dates slipped earlier: Christmas in the fall. The Julian Calendar has a negative error so Christmas moved later and later through the winter and into the spring.

- By 1582 the drift of the Julian Calendar amounted to about 10 days and was quite noticeable. So Pope Gregory XIII decreed that 10 days were to be skipped and instituted the Gregorian Calendar. According to this calendar leap years usually occur every 4th year but are skipped every 100 years unless the year is divisible by 400. 400, 800, 1200, 1600, and 2000 ARE leap years but 100, 200, 300, 500 . are not. Thus 2000 is a leap year but 1900 and 2100 are not. A four-century period will be missing 3 of its 100 Julian leap years, leaving 97. So the average year has  $365 + 97/400 = 365.2425$  days.

Many countries resisted adapting this calendar right up to the twentieth century to avoid any appearance of accepting the religious authority of the Pope, but it is now in universal civil use.

**Analysis:** Table 1 shows errors for each calendar discussed. Error ratios can be multiplied by 100 to give error percentages. The error percentage is quite small even for the 365-day calendar (about 0.07%).

$$\text{error ratio} = (\text{true} - \text{approx}) / \text{true}$$

$$= (1 - \text{approx} / \text{true})$$

**Table 1**

	<b>Average Year (Days)</b>	<b>Error Per Year (Days)</b>	<b>Error Ratio</b>	<b>Cycle Period</b>	<b>1-Day Shift</b>	<b>100-year Error (Days)</b>
365	365.0000	0.2422	6.63x10 <sup>-4</sup>	1,508	4.129	24.22
Julian	365.2500	-0.0078	2.14x10 <sup>-5</sup>	46,826	128.2	0.78
Gregorian	365.2425	-0.0003	8.2x10 <sup>-7</sup>	1,200,000	3,333.3...	0.03
4000	365.24225	-0.00005	1.4x10 <sup>-7</sup>	7,300,000	20,000	0.005
2 / 900	365.2422	-0.000022	6.1x10 <sup>-8</sup>	16,000,000	45,000	0.002
Jalaali	365.242424	-0.000224	6.1x10 <sup>-7</sup>	1,600,000	4,460	0.02
31 / 128	365.24218750	0.000125	3.4x10 <sup>-8</sup>	29,000,000	80,000	0.001

The cycle period is how long it takes for a date to cycle all the way back to where it started. The most interesting thing about this is that the Egyptians stuck with the 365-day year long enough for this to happen almost 3 times.

$$\text{cycle period} = \text{true} / |\text{error}|$$

$$= \text{true} / |\text{true} - \text{approx}|$$

$$= 1 / |1 - \text{approx} / \text{true}|$$

For more accurate calendars we may ask how long it takes for the error to accumulate just one day.

$$\text{day period} = 1 / |\text{error}| = 1 / |\text{true} - \text{approx}|$$

We also wonder if the shift in the seasons will be noticeable over the course of a long lifetime, say 100 years:

$$100\text{-year error} = |\text{error}| * 100 = 100 * |\text{true} - \text{approx}|$$

**Limits of Precision:** Before contemplating further corrections to the Gregorian Calendar we must consider how exact the value of 365.2422 is. The length of the average tropical year is now more precisely 365.24219 days but it varies somewhat from year to year and does not track the seasons precisely. Also, because of tiny orbital effects the average tropical year varies by about .00005 days per 1,000 years. Thus correcting any error of this magnitude is probably a waste of time.

**Design Tradeoffs:** The closer a calendar comes to having leap years 24.22% (0.2422) of the time the more accurate it is. But who wants an accurate calendar where it is difficult to answer questions like, "Is 2000 a leap year?" The Gregorian Calendar does well by being reasonably accurate (see analysis) while leaving determination of leap years fairly simple in almost all cases:

- If the year is not divisible by 4 then the year is not a leap year.
- If it is divisible by 4 it is definitely a leap year unless it ends in "00".
- Once every 100 years things get messier, but at least it is only once every 100 years.

**Optional Material 1 -- Refining the Gregorian Calendar:**

A final tweak has been proposed for the Gregorian system: removing leap years every 4000 years. Then 4000, 8000, 12000 etc. would NOT be leap years. A four-millennium period that would otherwise have had 970 leap years then has 969 leap years. The average year would then have 365 + 969/4000 = 365.24225 days.

Alternatively, the 400-year correction could be effectively changed to a 450-year correction with a pattern of 2 century-ending leap years per 900 years. This is what the "Revised Julian Calendar" of the Eastern Orthodox church does. 365 + 218/900 = 365.242222. They have chosen which 2 centuries those are so that this calendar and the Gregorian Calendar agree during the period from 1600 to 2800.

### Optional Material 2 -- Other Calendar Patterns:

Since the ideal interval between leap years is just over four years, it may be more sensible to look for a system where the interval is occasionally 5 years, never eight. The error in the Julian calendar was one day every 128.2 years. This is almost exactly 1/4 day every 32 years. So a pattern with 8 leap years in 33 years would be very accurate.  $8/33 = 0.242424\dots$ , an error of just .000224 days per year. This is more accurate than the Gregorian Calendar with a much shorter period, but is not convenient because it is not easy to figure out where a given year (say 2001) falls in the 33-year cycle. The Iranian Jalaali Calendar uses this system.

What about other patterns? The Gregorian Calendar repeats with a period of 400 years. What about patterns that recur every 2, 4, 8, 16,... years? We can investigate these by taking the fraction 0.2422 and repeatedly doubling it. We have a new pattern whenever our number rounds to an odd number:

$0.2422 * 2^{(n-1)}$	$2^{(n-1)}$	Fraction	Decimal Fraction	
0.2422	1			
0.4844	2			
0.9688	4	1/4	0.25	Julian Calendar
1.9376	8			
3.8752	16			
7.7504	32			
15.5008	64			
31.0016	128 *	31/128	0.2421875	

\* Skip one leap year every 128 years. More accurate than Gregorian Calendar.

### Related Exercises:

- There are many other columns that could be added to Table 1, including:
  - Error in hours, minutes and seconds per year.
  - Percent error.
  - How long it takes for a season to be shifted into the next season (1/4 or 1/3 or 1/2 year are all reasonable to use for the length of a season.) When would crops falter if a "bad" calendar was used for timing sowing and harvest? I'd guess 10-20 days shift.
- If your students can program computers then an easy exercise is to write a program that accepts a year number as input and tells whether it is a leap year or not. A harder exercise is to write a program that displays the error after each year. ( $0.2422 * \text{year} - \text{number of leap years}$ ).

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