Curriculum Topic Benchmarks: M1.4.1, M2.4.6, M3.4.3, M4.4.6, M4.4.8, M5.4.17
Grade Level: High School [9-12]
Subject Keywords: Area, Estimation, Proportion, Geometry, Surface Area, Scaling, Measurement
Rating: Moderate

Estimating Planar Areas Using Analogue Methods

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Obtaining the areas of oddly shaped, planar objects can be difficult.

Purpose: To determine the areas of simple and complex planar figures using measurement of mass and proportional constructs.

Materials

- Poster board or oak tag
- Scissors or other precise cutting tool
- Ruler
- Sharp pencil
- Right angle
- Analytical balance (0.1 milligram readability preferable – suggestions for working with less precise weighing tools are included below)

Procedure
Using a sharpened pencil, ruler and right angle, draw a square on the poster board or oak tag.
Cut out the square precisely. The size of the square should fit ***inside*** the weighing pan (for the examples below, we used a Mettler AB204-S analytical balance).
Draw and cut out other shapes of interest as suggested by the illustration below.

Turn on balance and allow it to warm up according to its specifications. Use the square as the standard. Have the students calculate its area, and determine its mass as precisely as possible.

Then weigh each shape and record each mass as well. Infer the area of each figure using the proportion:

$$\frac{\text{Area of Square}}{\text{Weight of Square}} = \frac{\text{Area of Shape}}{\text{Weight of Shape}}$$

You might try this first with a circle. Have the students calculate the area of the circle, and then determine it experimentally using the square as the standard. Do the same with other simple shapes, building confidence in the technique. Then experimentally derive the areas of some shapes that cannot be calculated simply.

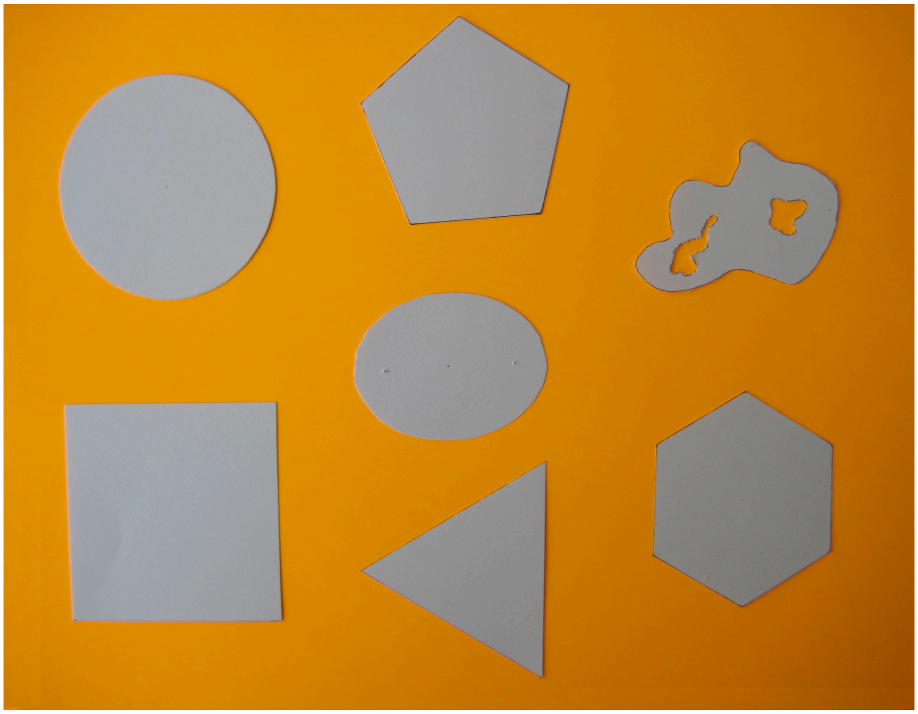

Fig. 1 Circle: diameter=2.50 inches, mass=0.5920 grams; Square: side=2.50 inches, mass=0.7329 grams; Regular Pentagon: side=1.50 inches, mass=0.4551 grams; Ellipse: semi-major axis=1.125 inches, semi-minor axis=0.8282 inches, mass=0.3385; Equilateral Triangle: side=2.50 inches, mass=0.3203 grams; Irregular: extreme dimensions=2.625 inches, 1.75 inches, weight=0.2819 grams; Regular Hexagon: side=1.25 inches, mass=0.4844 grams. Balance precision = 0.1 milligram.

Results

Using the simple proportion, we inferred the areas shown in table 1a. The balance we used can resolve mass to within 10^{-4} grams, which is nominally 0.045% of the mass of a piece of paper. This is a very small contribution to the overall error.

Shape	*True Area*	*Inferred Area*	*% Error*
Circle	4.9087	5.0484	2.8459
Square	6.2500	-	-
Pentagon	3.8711	3.8810	0.2554
Ellipse	2.9268	2.8866	1.3730
Triangle	2.7063	2.7314	0.9280
Irregular	-	2.4040	-
Hexagon	4.0595	4.1300	1.7576

Table 1a) Summary of the shapes used in calculations of areas
Balance precision = 0.1 milligram.

Results derived using a less-precise balance, having a precision of 1 milligram (masses of shapes not shown), are displayed in table 1b. In this case, the reduced scale resolution affects the inferred areas only slightly. For less precise balances, ways to reduce the percent error include: (1) using larger shapes, (2) stacking multiple cutouts of the same figure and/or (3) using thicker, solid cardboard sheets.

Shape	*True Area*	*Inferred Area*	*% Error*
Circle	4.9087	5.056	3.000
Square	6.2500	-	-
Pentagon	3.8711	3.896	0.643
Ellipse	2.9268	2.894	1.121
Triangle	2.7063	2.737	1.134
Irregular	-	2.406	-
Hexagon	4.0595	4.149	2.205

Table 1b) Summary of the shapes used in calculations of areas. Balance precision = 1.0 milligram.

Discussion

The inferred areas can be seen to agree well with the theoretical values. The square is used as a standard from which all other areas are derived because its simple geometry can be cut out very precisely, hence its area can best approximate the theoretical value that is given by the square of the length of a side. The theoretical area of the irregular figure is unknown because it is the object of this lesson to demonstrate how one might simply calculate the area of a complex shape that may defy analytic tractability.

To understand the sources of error, thickness measurements of each shape were taken randomly using a micrometer capable of measuring to within 50×10^{-6} inches. The histogram in figure 2 shows the oak tag mean thickness to be 9.600×10^{-3} inches with a standard deviation of 1.2223×10^{-4} inches resulting in a thickness error contribution of approximately 1.27%, 68.3% of the time. Thus, the source of error due to variations in thickness is small relative to errors made from cutting out the figures.

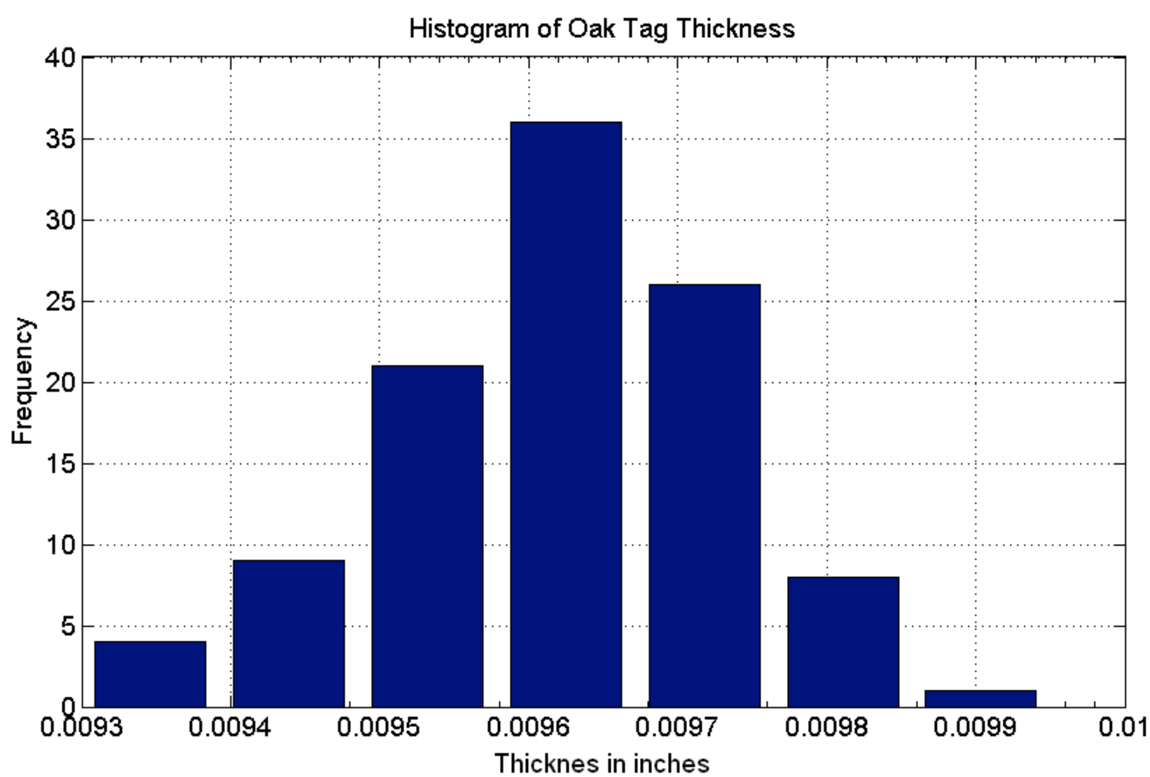

Fig. 2 Histogram depicting oak tag thickness variations.

Conclusion

A method for calculating simple and complex planar areas is described in this lesson. Using a balance to weigh shapes of known areas, an area-to-weight ratio is formed. The unknown area of a second shape of interest is calculated by multiplying its weight by the aforementioned ratio. This procedure is useful because of its generality: areas of regions that are not analytically tractable can be readily estimated. The accuracy is limited mainly by the precision of cutting the shape's boundary.

Extension – Scaling Up and Down

To expand or contract an area, it is observed that the ratio of the inferred area of interest (A_x) to the reference area (i.e., the square) (A_R) is invariant with respect to scale. Let $R= A_x/ A_R$, where $A_R=L^2$. If the length L of the square is rescaled to λL, the new area $A_R^{'}=(\lambda L)^2= \lambda^2 A_R$. Since the ratio of areas is an invariant, it follows that the rescaled area of interest, $A_x^{'}=R\,\lambda^2 A_R$. If $\lambda>1$, the area is expanded. If $\lambda<1$ the area is contracted. In the examples above, the irregular figure has ratio R=2.404/6.250=0.385 and let it be required to apply the rescaling $\lambda=4$ miles/inch, as given by a map. Then, the rescaled area will expand to:

$A_x^{'}=0.385\times(4 miles / inch)^2 \times 6.25 inch^2$ =38.5 square miles.

Practicality

The method can be applied to any problem requiring planar area calculations. Examples span the gamut from theoretical calculations of integrals, to practical calculations of areas of regions measured by Google-Earth (e.g., city blocks, school yards, lakes, parks...etc), provided the curvature of the earth can be ignored. Google-Earth is especially useful since it provides a "ruler" to measure distances on a map. Calculations of rescaled areas are performed using the transformations indicated above. One of us (PG) has applied this technique to the calculation of complex planar areas that were used in a technical, peer-reviewed paper[1]. It was a timesaving method that yielded very good accuracy. Finally, environmental humidity will exert minimal influence on the calculation of areas if the same material is used to create all the shapes.

Note

There are other methods that can be used to infer areas that do not require the use of a balance. For example, it is possible to use Adobe Photoshop to process a digitized image as is demonstrated in the YouTube video: http://www.youtube.com/watch?v=E3O-V6WLw0g

That method, also based on proportional constructs, was applied to the figures above with varying degrees of success, dictated by the quality of the image as measured in terms of the uniformity of its surface brightness and the presence of shadows at its edges.

Reference

1. Gabriel, P., H. W. Barker, D. O'Brien, N. Ferlay and G. L. Stephens. 2009. Statistical Approaches to Plane-Parallel Error Identification and Retrievals of Optical and Microphysical Properties in 3D Clouds Part 1: Bayesian Inference. *J. Geophys. Res.*, **114**, D06207, doi: 10.1029/2008JD011005.