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Geometric Addition + Logarithms = Geometric Multiplication

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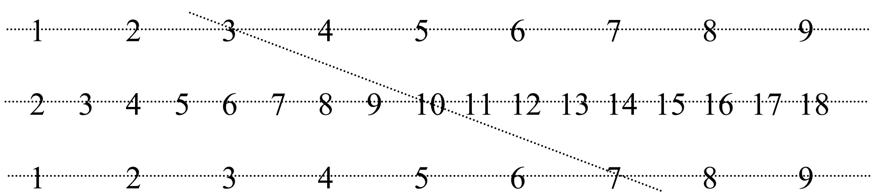
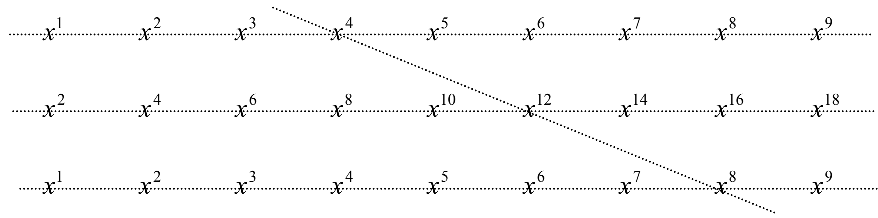
Mathematicians often argue that anything which can be represented numerically or algebraically can also be represented geometrically. This is perhaps true even to the extent that simple numeric calculations can be demonstrated geometrically. The following example illustrates one such geometric process of addition.

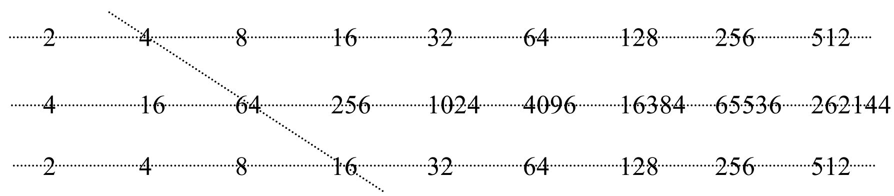


A transversal passing through any two points on the exterior lines passes through the middle line at the sum (as long as the lines are evenly spaced). The example above represents the expression 3 + 7 = 10. Let us explore for a moment how this works. Because the middle line (where we find the sum) is evenly spaced between the exterior lines (where the two addends are located), the *distance* of the middle line to the point where the result is found is essentially the *average* of the distances along the outer lines. The average of 3 and 7 is (3 + 7) / 2 or 5. Because the sum of 3 and 7 is twice their average, the points on the middle line have numeric values twice as large as on the outer lines. Subtraction then can be performed on the same graph since any addition problem can be transformed into a subtraction problem. A number on either the upper or lower line can subtracted from a number on the middle line. The same line shown in the example can be used in reverse to find that 10 - 3 = 7 or 10 - 7 = 3.

Can a similar geometric technique be used for multiplication? Of course! You may recall from algebra that $x^n \cdot x^k = x^{n+k}$. This algebraic property provides a process for multiplying polynomials, and ultimately this simple exponent rule acts as an introduction to the properties of logarithms. $x^n \cdot x^k = x^{n+k}$ translates to logarithmic form as: $\log_x(nk) = \log_x n + \log_x k$, which is essentially an addition process. The use of logarithms is what allows us to perform multiplication processes by using addition. Because logarithms provide a geometric representation of multiplication, we can use number lines similar to those that work for addition in the first example, but the *distances* along the lines now correspond to *logarithms* of the numbers they represent. Consequently, if similar number lines are expressed as increasing powers of some base, the same addition property should hold true as in the first example. Consider the example $x^4 \cdot x^8 = x^{12}$:



If ‘x’ from the previous example is now assigned a value greater than one (the value for ‘x’ must be greater than one since 1 raised to any power will never increase), a geometric multiplication chart is the result. Although any base number other than 1 could be used, for ease of calculation, we will let $x = 2$ and evaluate each resulting exponent along the number line. The example below demonstrates. $4 \cdot 16 = 64$



It is of course inconvenient that as the multiplied factors become larger, it becomes more difficult to determine exactly what value is represented as the product on the middle line. Note however that the transversal can cross the number lines anywhere (not just on the labeled integers) and the resulting product will be accurate providing the user can interpolate between the larger values with some accuracy. And of course, division problems can be addressed in the same fashion as the subtraction in the first example. Starting with the dividend in the middle and selecting the divisor from one side, the quotient can be found on the opposite exterior line. Furthermore, with some accurate interpolation, square roots can be obtained by drawing vertical transversals anywhere along the scale.

The concepts related to exponents and logarithms are the foundation of this kind of geometric multiplication, just as they are for the old fashion slide rule. The problem with a slide rule is that it must slide. Two pieces are needed to multiply and divide, and so it can become impractical or cumbersome if one is not practiced. The extensions and applications of number lines however are very simple and virtually limitless.

For more information and great teaching ideas on logarithms refer to PUMAS examples 06_01_97_1: “Just What is a Logarithm, Anyway?”, 06_05_99_1: “Learning from Slide Rules,” and 12_30_02_1: “Logarithms: Taking the Curve Out.”