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Isoperimetric Geometry

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The **isoperimetric theorem** states that: "Among all shapes with an equal area, the circle will be characterized by the smallest perimeter" which is equivalent to "Among all shapes with equal perimeter, the circle will be characterized by the largest area." The theorem's name derives from three Greek words: 'isos' meaning 'same', 'peri' meaning 'around' and 'metron' meaning 'measure'. A perimeter (= 'peri' + 'metron') is the arc length along the boundary of a closed two-dimensional region (= a planar shape). So, the theorem deals with shapes that have equal perimeters.

History of the theorem

The theorem was already known 900 years BC. An application is found in the story of Dido, as described in Virgil's *Aneid*. Dido was a princess on the run from Tyrus (nowadays Lebanon). She arrived in North Africa, at a site later known as Carthage (nowadays Tunisia). She wanted to buy some land from the local ruler, King Jambas. They agreed that she could buy all the land she could enclose within a bull's hide. Consequently, Dido had the bull's hide cut into small strips and had the strips stitched together. Let's try to estimate the length of the ribbon used by the princess. We assume that the bull's body is shaped like a cylinder, with a diameter of 100 cm and a length of 200 cm. Since the width of the strips used by Dido did not exceed 0.5 cm, we can estimate the length for a fixed width of 0.5 cm. Using our cylinder-shaped bull, we find the ribbon's length is 1,257 m, which lies within the range of 1,000 and 2,000 m as reported by Virgil.

Dido put the ribbon on the ground in a way that it would enclose the maximum area. So Dido had to solve the isoperimetric theorem! According to the story, Princess Dido solved the problem, and acquired 8 to 32 hectares of land. [A hectare is 10,000 m^2.] Let's do the exercise for a ribbon of length 1,257 m. If the ribbon is placed on the ground in a circle, it will enclose 12.6 hectares. If the ribbon is arranged in a square, it encloses 9.9 hectares. As rectangles with sides 200x428.5 or 100x528.5 m, the corresponding areas are 8.6 and 5.3 hectares, respectively. However, Princess Dido used one more trick. Virgil relates that Dido connected the end points of the ribbon to a straight segment of the Mediterranean coast, so the ribbon formed the shape of a semicircle. In this way, even a larger area was enclosed, since the sea coast length was added to the ribbon length. For our ribbon of 1,257 m, this would generate an area of 25.1 hectares, which lies within the reported range.

Some math …

Let's demonstrate the theorem for a rectangle and a circle (Figure 1). This is not a real mathematical proof, but elaboration of this example adequately illustrates the theorem. The area of the rectangle (*AR*) is calculated as ($a\times b$); the area of the circle (*AC*) is calculated as ($\pi\times r^2$), with π equal to 3.1416. If the shapes have equal areas ($AR=AC$), the following equation applies: $(a\times b)=(\pi\times r^2)$. Solving this equation for r, one finds $r=\sqrt{(a\times b/\pi)}$. The perimeter of a circle (*PC*) is calculated as $(2\times\pi\times r)$. Using a and b instead of r, we find $PC=2\sqrt{(a\times b\times\pi)}$.

To prove the isoperimetric theorem, we have to demonstrate that *PC* is smaller than the perimeter of the rectangle (*PR*), which equals $2\times(a+b)$. Hence the inequality to solve is $PC<PR$, or, $2\sqrt{(a\times b\times\pi)}<2\times(a+b)$. Recall from other math classes that for $x,y>0$, if $x<y$, then $x^2<y^2$. If we apply this on our inequality, we find $(a\times b\times\pi)<(a+b)^2$. The inequality can be rewritten $a^2+b^2>(\pi-2)\times a\times b$. Remember also from math classes that if $x>(z\times y)$ and ($v<z$), then $x>(v\times y)$. This applies for $v,x,y,z>0$. Considering $2>(\pi-2)$, the inequality is proven if $a^2+b^2>2\times(a\times b)$. Now, recall that we compare a rectangle with a circle (Figure 1). For the rectangle let $a>b$ (otherwise it would be a square). We can write this as $a=b+k$, with $k>0$. Substitution of b by $(a-k)$ gives $a^2+(a-k)^2>2\times a\times(a-k)$. Elaborating this inequality gives $k^2>0$, which is valid for all $k>0$. So, the inequality is valid. The circle's perimeter is smaller that that of a rectangle having equal area.

Now let's prove the theorem for the alternative formulation. It can be written: $AR<AC$ if $PC=PR$. Consequently, the starting point is $(2\times\pi\times r)=2\times(a+b)$. This generates $r=(a+b)/\pi$. Rewriting *AC* using a and b, the following inequality should be proven: $(a+b)^2/\pi>(a\times b)$. This inequality can be simplified into $(a+b)^2>(\pi\times a\times b)$, which was proven earlier. The rectangle and circle have equal perimeters, and the circle's area is larger.

Exercise

Consider the shapes in Figure 2. Note that $a=2\times b$ for the rectangle and $d=2\times h$ for the triangle.

(1) Calculate the perimeters of the shapes, assuming every shape has an area of 1 m^2.

(2) Calculate the area of the shapes, assuming every shape has a perimeter of 4 m.

(3) Do the calculations confirm the isoperimetric theorem?

Answers

(1) Rectangle: 4.24 m; Circle: 3.54 m; Triangle: 4.83 m; Square: 4.00 m.

(2) Rectangle: 0.88 m^2; Circle: 1.27 m^2; Triangle: 0.69 m^2; Square: 1.00 m^2

(3) The calculations illustrate the isoperimetric theorem. When all shapes have equal areas, the circle has the smallest perimeter. When all shapes have equal perimeters, the circle encloses the largest area. In both cases, the circle is followed by the square. Unlike the triangle and the rectangle in the example, the square and circle can be described by single parameters (r for the circle, c for the square). Circles and squares are 'isodiametric' shapes ('isos' = 'equal', 'dia' = 'through', 'metron' = 'measure'). The circle's characteristic length is the same when measured in all directions through the center (the diameter), whereas the square's characteristic length occurs in only two perpendicular directions.

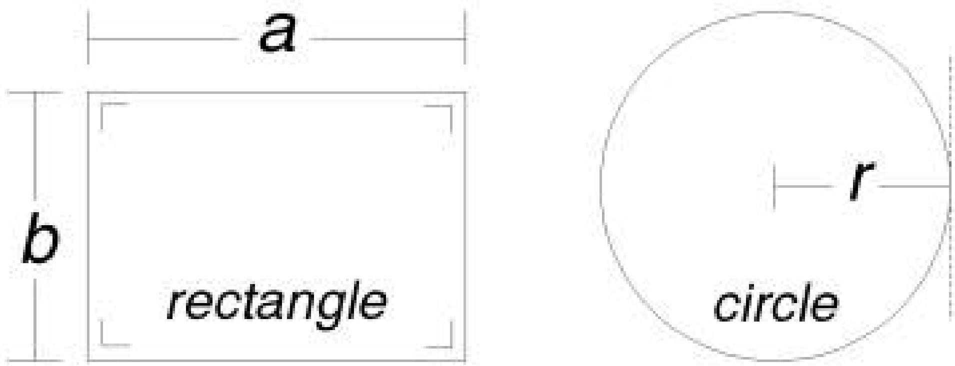

Figure 1

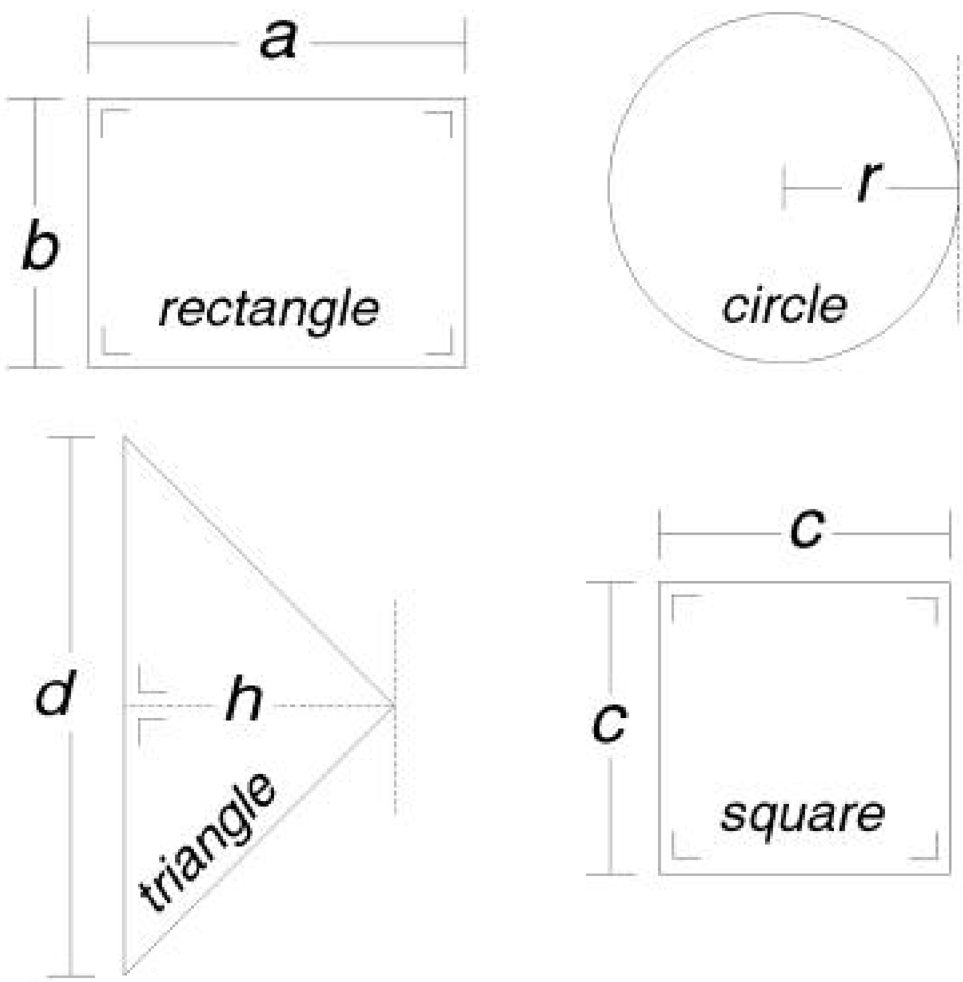

Figure 2